Structure and Interpretation of Classical Mechanics
Problem Set 3

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March 13, 2017

Exercise 1.44: Double pendulum

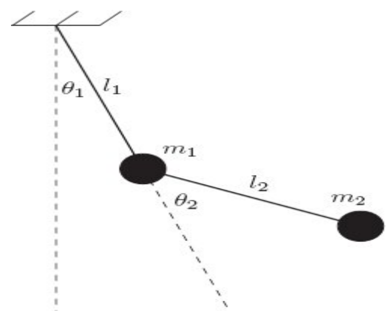

Figure 1: The double pendulum is pinned in two joints so that its members are free to move in a plane.

Consider the ideal double pendulum shown in figure 1.

a. Formulate a Lagrangian to describe the dynamics. Derive the equations of motion in terms of the given angles θ_1 and θ_2. Put the equations into a form appropriate for numerical integration. Assume the following system parameters (with $g = 9.8$ m/s^2):

$$l_1 = 1.0 \text{ m} \qquad m_1 = 1.0 \text{ kg}$$
$$l_2 = 0.9 \text{ m} \qquad m_2 = 3.0 \text{ kg}$$

We begin by choosing our generalized coordinates to be $q = (\theta_1, \theta_2)$. Therefore, we can write the rectangular coordinates of the two bobs in terms of θ_1 and θ_2:

$$\mathbf{x}_1 = f_1(t, q) = (l_1 \sin\theta_1, -l_1 \cos\theta_1) \quad \text{and} \quad \mathbf{x}_2 = f_2(t, q) = (l_1 \sin\theta_1 + l_2 \sin\theta_2, -l_1\cos\theta_1 - l_2\cos\theta_2)$$

Next, we compute the generalized velocities:

$$\begin{aligned}
\mathbf{v}_1 &= \partial_0 f_1(t,q) + \partial_1 f_1(t,q)v \\
&= 0 + [\partial_{1,0} f_1(t,q), \partial_{1,1} f_1(t,q)](\dot\theta_1, \dot\theta_2) \\
&= [[l_1\cos\theta_1, l_1\sin\theta_1], 0](\dot\theta_1, \dot\theta_2) \\
&= [l_1\dot\theta_1\cos\theta_1, l_1\dot\theta_1\sin\theta_1]
\end{aligned}$$

$$\begin{aligned}
\mathbf{v}_2 &= \partial_0 f_2(t,q) + \partial_1 f_2(t,q)v \\
&= 0 + [\partial_{1,0} f_2(t,q), \partial_{1,1} f_2(t,q)](\dot\theta_1, \dot\theta_2) \\
&= [[l_1\cos\theta_1, l_1\sin\theta_1], [l_2\cos\theta_2, l_2\sin\theta_2]](\dot\theta_1, \dot\theta_2) \\
&= \dot\theta_1[l_1\cos\theta_1, l_1\sin\theta_1] + \dot\theta_2[l_2\cos\theta_2, l_2\sin\theta_2]] \\
&= [l_1\dot\theta_1\cos\theta_1, l_1\dot\theta_1\sin\theta_1] + [l_2 dot\theta_2\cos\theta_2, l_2\dot\theta_2\sin\theta_2]] \\
&= [l_1\dot\theta_1\cos\theta_1 + l_2\dot\theta_2\cos\theta_2, l_1\dot\theta_1\sin\theta_1 + l_2\dot\theta_2\sin\theta_2]
\end{aligned}$$

Using the generalized velocities, we can write the kinetic energy of the system as

$$\begin{aligned}
T &= \frac{1}{2}m_1\mathbf{v}_1^2 + \frac{1}{2}m_2\mathbf{v}_2^2 \\
&= \frac{1}{2}m_1 l_1^2\dot\theta_1^2(\sin^2\theta_1 + \cos^2\theta_1) + \frac{1}{2}m_2[(\dot\theta_1 l_1\cos\theta_1 + \dot\theta_2 l_2\cos\theta_2)^2 + (\dot\theta_1 l_1\sin\theta_1 + \dot\theta_2 l_2 \sin\theta_2)^2] \\
&= \frac{1}{2}m_1 l_1^2\dot\theta_1^2 + \frac{1}{2}m_2[l_1^2\dot\theta_1^2\cos^2\theta_1 + 2l_1l_2\dot\theta_1\dot\theta_2\cos\theta_1\cos\theta_2 + l_2^2\dot\theta_2^2\cos^2\theta_2 + l_1^2\dot\theta_1^2\sin^2\theta_1 + 2l_1l_2\dot\theta_1\dot\theta_2\sin\theta_1\sin\theta_2 + l_2^2\dot\theta_2^2\sin^2\theta_2] \\
&= \frac{1}{2}m_1 l_1^2\dot\theta_1^2 + \frac{1}{2}m_2[l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2l_1l_2\dot\theta_1\dot\theta_2(\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2)] \\
&= \frac{1}{2}m_1 l_1^2\dot\theta_1^2 + \frac{1}{2}m_2[l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2l_1l_2\dot\theta_1\dot\theta_2(\frac{1}{2}[\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2])] \\
&= \frac{1}{2}m_1 l_1^2\dot\theta_1^2 + \frac{1}{2}m_2[l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + l_1l_2\dot\theta_1\dot\theta_2(\cos(\theta_1-\theta_2) - \cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2) + \cos(\theta_1+\theta_2))] \\
&= \frac{1}{2}m_1 l_1^2\dot\theta_1^2 + \frac{1}{2}m_2[l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2l_1l_2\dot\theta_1\dot\theta_2\cos(\theta_1-\theta_2)].
\end{aligned}$$

Furthermore, the potential energy of the system is given by:

$$
\begin{aligned}
V &= m_1 g y_1 + m_2 g y_2 \\
&= -m_1 l_1 g \cos\theta_1 + m_2 g(-l_1 \cos\theta_1 - l_2 \cos\theta_2) \\
&= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2
\end{aligned}
$$

Therefore, the Lagrangian for the double pendulum is

$$
\begin{aligned}
L &= T - V \\
&= \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2
\end{aligned}
$$

With the Lagrangian in hand, SCMUTIL's `Lagrange-equations` procedure allows the Lagrange equations with relative ease. We find that the residuals are given by:

$$
\begin{bmatrix}
-l_1 l_2 m_2 \sin(\theta_2(t) - \theta_1(t))(D\theta_2(t))^2 + l_1 l_2 m_2 \cos(\theta_2(t) - \theta_1(t)) D^2\theta_2(t) + g l_1 m_1 \sin(\theta_1(t)) + g l_1 m_2 \sin(\theta_1(t)) + {l_1}^2 m_1 D^2\theta_1(t) + {l_1}^2 m_2 D^2\theta_1(t) \\
l_1 l_2 m_2 \sin(\theta_2(t) - \theta_1(t))(D\theta_1(t))^2 + l_1 l_2 m_2 \cos(\theta_2(t) - \theta_1(t)) D^2\theta_1(t) + g l_2 m_2 \sin(\theta_2(t)) + {l_2}^2 m_2 D^2\theta_2(t)
\end{bmatrix}
$$

b. Prepare graphs showing the behavior of each angle as a function of time when the system is started with the following initial conditions:

$$
\begin{aligned}
\theta_1(0) &= \pi/2 \text{ rad} & \qquad \dot{\theta}_1(0) &= 0 \text{ rad s}^{-1} \\
\theta_2(0) &= \pi \text{ rad} & \qquad \dot{\theta}_2(0) &= 0 \text{ rad s}^{-1}
\end{aligned}
$$

Make the graphs extend to 50 seconds.

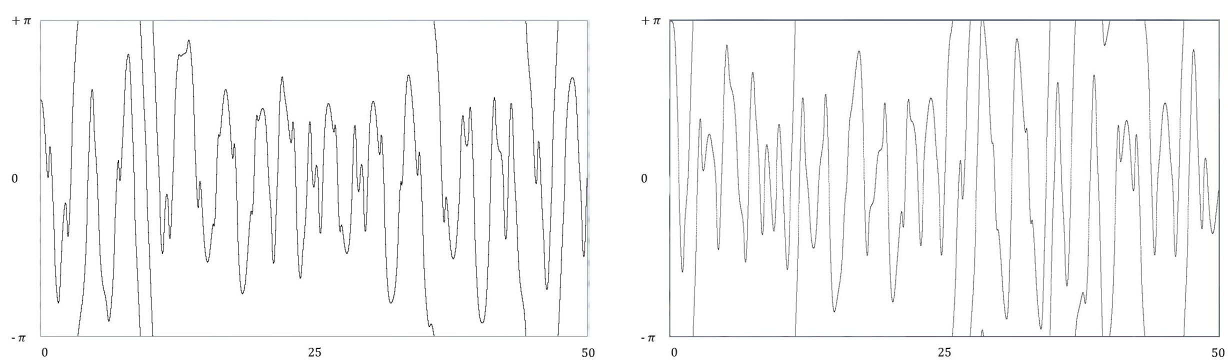

Figure 2: Orbit of the double pendulum. The angles θ_1 (left) and θ_2 (right) are plotted against time with initial conditions listed above.

c. Make a graph of the behavior of the energy of your system as a function of time. The energy should be conserved. How good is the conservation you obtained?

(see figure 3)

d. Make a new Lagrangian, for two identical uncoupled double pendulums. (Both pendulums should have the same masses and lengths.) Your new Lagrangian should have four degrees of freedom. Give initial conditions for one pendulum to be the same as in the experiment of part b and give initial conditions for the other pendulum with the m_2 bob 10^{10} m higher than before. The motions of the two pendulums will diverge as time progresses. Plot the logarithm of the absolute value of the difference of the positions of the m2 bobs in your two pendulums against the time. What do you see?

Figure 3: Energy of system as a function of time. The energy is indeed conserved to conserved to the point where numerical instability is the main driver of any changes in energy over time.

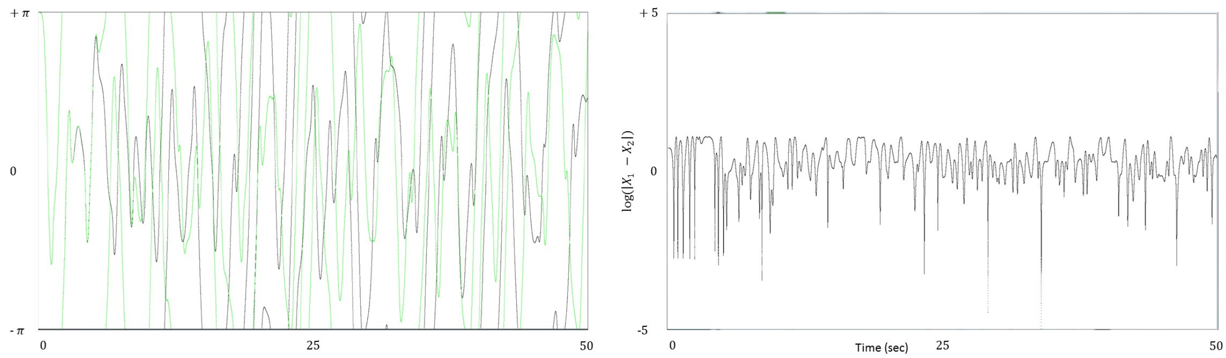

Figure 4: The motion of two independent double pendulums diverge over time, even with essentially negligible differences in initial conditions. (Left) Angular position of two independent double pendulums. Their trajectories begin to diverge after about 6 seconds with only 10^{-10} m difference in the initial starting heights. (Right) Log of the absolute euclidean distance between the m_2 bobs.

e. Repeat the previous comparison, but this time use the base initial conditions:

$$\theta_1(0) = \pi/2 \text{ rad} \qquad \dot{\theta}_1(0) = 0 \text{ rad s}^{-1}$$
$$\theta_2(0) = 0 \text{ rad} \qquad \dot{\theta}_2(0) = 0 \text{ rad s}^{-1}$$

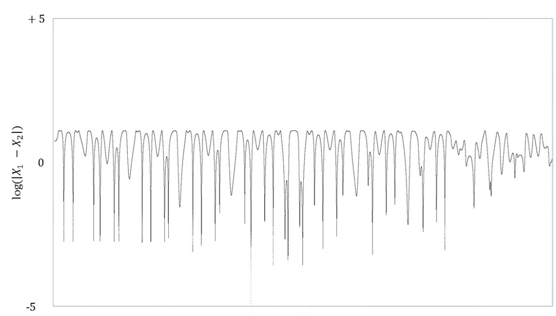

Figure 5: Log of the absolute euclidean distance between the m_2 bobs as a function of time.

Comparing the results from (e) and (d), it appears that the initial condition of $\theta_2(0) = 0$ rad has the effect of producing more regular (less "chaotic") behavior, which is why we observe more frequent downward spikes in figure 5 (when the two bobs come within 1 meter of each other. This intuitively makes sense because this configuration doesn't "engage" the second arm of pendulum initially.